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Massless BTZ black holes in minisuperspace

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Abstract: We study aspects of the propagation of strings on BTZ black holes. After performing a careful analysis of the global spacetime structure of generic BTZ black holes, and its relation to the geometry of the $\mathrm{SL}(2,\mathbb{R})$ group manifold, we focus on the simplest case of the massless BTZ black hole. We study the $\mathrm{SL}(2,\mathbb{R})$ Wess-Zumino-Witten model in the worldsheet minisuperspace limit, taking into account special features associated to the lorentzian signature of spacetime. We analyse the two- and three-point functions in the pointparticle limit. To lay bare the underlying group structure of the correlation functions, we derive new results on Clebsch-Gordan coefficients for $\mathrm{SL}(2,\mathbb{R})$ in a parabolic basis. We comment on the application of our results to string theory in singular time-dependent orbifolds, and to a lorentzian version of the AdS/CFT correspondence.

Keywords: Black Holes in String Theory, AdS-CFT and dS-CFT Correspondence.

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1. Introduction

The AdS/CFT correspondence gave rise to the hope that we might understand quantum gravity in black hole backgrounds better through their field theoretic holographic dual. Since string theory is a consistent theory of quantum gravity, we can study strings on black hole backgrounds that are asymptotically AdS to try and realize that hope. In particular, it is known that BTZ-black holes, which are asymptotically AdS_3, can be thought of as orbifolded WZW models. They are stringy backgrounds in which the holographic duality could be under sufficient control to understand features of strings on singular backgrounds. That's a first reason to study these backgrounds in more detail.

Recently, singular time-dependent orbifold backgrounds of string theory have been investigated (see e.g. [1, 2, 3] and references thereto). It was argued that, close to the singularity (and in the limit of vanishing cosmological constant), a massless BTZ black hole resembles a parabolic orbifold of Minkowski space [2, 4]. To shed further light on whether string theory can deal with the parabolic singularity or not (see e.g. [5]–[8]), it will be very interesting to see whether the WZW model of the massless BTZ black hole is under good control. Moreover, since it is asymptotically AdS_3, it can perhaps provide us with a holographic interpretation of the troublesome features of strings on these singular backgrounds.

One more reason to believe that the massless BTZ black hole background is important in this context, is the fact that it is supposed to describe the same boundary CFT_2 as the AdS_3 background, with different boundary conditions on the fermions [9, 10]. When the boundary CFT_2 has $N = 2$ supersymmetry in two dimensions, the boundary field theories are related by spectral flow. That suggests that at least the boundary conformal field theory corresponding to the massless BTZ black hole background is well-defined (since the boundary conformal field theory dual to AdS_3 backgrounds is). Thus, a more careful study of this background and its holographic properties should shed light on the nature of parabolic orbifolds and their stability.

A formal reason to believe that the orbifolded WZW theories might be manageable, is the underlying group structure. Therefore, it is important to further analyse the connection between the structure of the (extended) BTZ spacetimes, and the underlying group manifold and we do so in section 2. In section 3, we concentrate on the massless BTZ black hole, with the motivations above in mind, and because it is the example with the simplest global structure. It is a background that is on the verge of developing a massive black hole with a non-trivial horizon, and as such seems very interesting to get to grips with.

In the course of our work we will develop some techniques that should be useful to better understand typically lorentzian aspects of the AdS/CFT correspondence.

2. Global properties

In this section we review some of the global properties of the BTZ black hole backgrounds [11, 12] and stress some features that are important from a group theoretic perspective. That is crucial to us since we would like to study the Wess-Zumino-Witten model

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in the BTZ black hole background. Most of the properties that we list are known (see e.g. [10, 13]–[18]), but some are not, and they are needed for a good understanding of our treatment of the massless black hole background.

First of all, it is important to realize that the energy and angular momentum operator in BTZ black hole backgrounds generate two one-parameter subgroups of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ whose character depends on the particular choice of background. The one-parameter subgroups are elliptic (two-dimensional rotations) for ordinary AdS_3, they are hyperbolic (two-dimensional boosts) for the generic BTZ black hole, parabolic (shifts) for the massless BTZ black hole and one hyperbolic and one parabolic one-parameter subgroup for the extremal BTZ black hole. Those facts suggest a preferred choice of coordinates and a preferred basis for the $\mathrm{SL}(2,\mathbb{R})$ group manifold. We make use of those preferred coordinate choices in the following.

In [19] a nice analysis of the structure of the boundary for a generic BTZ black hole was presented, and its consequences for the structure of the boundary conformal field theory were analysed. The analysis made use of the natural group parametrisation. We want to analyze similar, but much simpler properties of the massless BTZ black hole background. To that end, it will be useful to describe some global features of the BTZ spacetimes, and their interpretation in terms of the $\mathrm{SL}(2,\mathbb{R})$ group manifold.

Since the main part of our paper is on the massless BTZ black hole, the reader can skip to subsection 2.4 on that particular case when she is not interested in the following generalities.

2.1 AdS_3

For the well-studied AdS_3 background, we choose global coordinates. To facilitate the diagonalisation of the action of elliptic subgroups on kernels (i.e. matrix elements) of $\mathrm{SL}(2,\mathbb{R})$ representations, we parametrise the group element in the following way:

$$g = e^{\frac{i}{2}(t+\phi)\sigma^2} e^{\rho\sigma^3} e^{\frac{i}{2}(t-\phi)\sigma^2} . \tag{2.1}$$

That is the appropriate parametrisation of the $\mathrm{SL}(2,\mathbb{R})$ WZW model for the description of string propagation on the covering, namely AdS_3. In $\mathrm{SL}(2,\mathbb{R})$ there is a closed timelike curve, which gives rise to a non-trivial first homotopoy group $\Pi_1(\mathrm{SL}(2,\mathbb{R})) = Z$. After unwinding the time direction t, we find the AdS_3 manifold in global coordinates:

$$ds^2 = -\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d\phi^2 . \tag{2.2}$$

Clearly, the boundary of the manifold is $R \times S^1$ and it is parametrized by (t,ϕ) at $\rho \to \infty$. The AdS_3/CFT_2 correspondence has been analysed in detail in this case (see e.g. [20]–[23]).

2.2 Generic BTZ black hole

The energy and angular momentum operators generate two hyperbolic one-parameter subgroups in $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ in the generic BTZ background. A parametrisation appropriate for diagonalising these operators is:[1]

$$g = e^{\frac{u}{2}\sigma^3}(-1)^{\epsilon_1}(i\sigma^2)^{\epsilon_2} p_i e^{\frac{v}{2}\sigma^3} \tag{2.3}$$

where p_i takes one of the forms

$$\begin{aligned} p_1 &= e^{\rho_1\sigma^1} \\ p_2 &= e^{i\rho_2\sigma^2} \end{aligned} \tag{2.4}$$

where $\rho_1 \in]-\infty, \infty[$ and $\rho_2 \in [-\pi/4, \pi/4[$. The structure of the boundary in this case has been analysed in some detail in [19]. We have little to add to their analysis, but we want to stress some of the global features of the BTZ spacetime, and how they relate to the WZW theory.

Discrete symmetries

First of all, we want to point out a few discrete symmetries of the WZW model that will turn out to be useful [14]. The following operations are symmetries of the CFT: the reflection $R : g \to -g$ and Bargmann's automorphism B that amounts to conjugation by σ_3. It acts on a group element $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ as $g \to \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$ Note that σ_3 is not an element of the group $\mathrm{SL}(2, \mathbb{R})$ such that this is an outer automorphism.

Let us now illustrate the action of these symmetries using the (un)extended BTZ spacetime. This will turn out not to be very fruitful, but will serve as a useful warm-up for what follows. By using the parametrisation introduced in (2.3), we can carve up the group manifold in 16 regions. First we distinguish eight bigger regions $\pm D_i$ where $+D_1$ corresponds to the choice $\epsilon_1 = 0 = \epsilon_2$ and p_1 in (2.3) and similarly for $i = 2$ and p_2. The two regions $+D_{3,4}$ similarly correspond to the choice $\epsilon_2 = 1$, $\epsilon_1 = 0$ and $p_{1,2}$. When $\epsilon_1 = 1$ we denote the regions $-D_i$ (with the same conventions as before for the index i). We can moreover split each of the eight regions $\pm D_i$ into two, creating sixteen patches. We split each $\pm D_i$ as $\pm D_i = \pm D_i^+ \cup \pm D_i^-$ according to the sign of ρ_j in (2.4).

Now, it is easily checked using the description in [12] of the geometry that the unextended BTZ spacetime (with mass M and angular momentum J),

$$ds^2 = -\left(-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}\right) dt^2 + \left(-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}\right)^{-1} dr^2 + r^2 \left(d\phi - \frac{J}{2r^2} dt\right)^2, \tag{2.5}$$

is equivalent to the group manifold modded out by the discrete subgroup generated by the two discrete symmetries R and B. (A fundamental domain of the group action is conveniently chosen to be for instance $\cup_i + D_i^+$.)

The extended BTZ spacetime [12] corresponds to copying the fundamental region an infinite number of times. It can be thought of as the original group manifold, where we take the covering with respect to the closed timelike curve. This essentially reintroduces the other regions D_i, and an infinite number of copies of these regions. That agrees (for instance) with the analysis presented in [19].

[1] Throughout, we ignore the extra patches corresponding to matrices with a zero. It would be interesting to clarify whether they play a role in holography.

The discrete symmetries will turn out to be more useful to us in the degenerate cases that follow.

2.3 Extremal black hole

The story already becomes different for the extremal black hole. We recall here a few different parametrisations of the extremal black hole background, firstly to make contact with the group theoretic description, secondly because they will turn out to be useful in analysing the bulk-to-boundary propagator.

The standard form of the metric for the extremal black hole (where $M = |J|$) is

$$ds^2 = -(-M + r^2)dt^2 + \left(\frac{M}{2r} - r\right)^{-2} dr^2 + r^2 d\phi^2 - J d\phi dt \,. \tag{2.6}$$

In the following we will choose $\mathrm{sign}(J) = 1$, without loss of generality. To simplify the description in terms of the natural group coordinates we perform several coordinate transformations. First, we introduce the coordinate $U^2 = r^2 - M/2$. In terms of the new coordinates the metric becomes:

$$ds^2 = U^2 \left(-dt^2 + d\phi^2\right) + \frac{M}{2}(dt - d\phi)^2 + \frac{dU^2}{U^2} \,. \tag{2.7}$$

These coordinates can be related to standard Poincare coordinates by the transformations:

$$\begin{aligned} \lambda_+ &= \frac{1}{2\pi T_+} e^{2\pi T_+(-\phi+t)} \\ \lambda_- &= -\phi - t - \frac{\pi T_+}{U^2} \\ z &= \frac{1}{U} e^{\pi T_+(-\phi+t)} \,, \end{aligned} \tag{2.8}$$

where the temperature T_+ is equal to $T_+ = \frac{r_+ + r_-}{2\pi} = \frac{r_+}{\pi} = \frac{1}{\pi}\sqrt{\frac{M}{2}}$ for the extremal black hole. We obtain the metric

$$ds^2 = \frac{1}{z^2}(dz^2 + d\lambda_+ d\lambda_-) \,. \tag{2.9}$$

Now we can compare to the appropriate $\mathrm{SL}(2,\mathbb{R})$ parametrisation:

$$g = (-1)^l e^{uL^-} e^{\rho L^+} e^{v\sigma^3} \,, \tag{2.10}$$

where σ^i are the Pauli matrices and $L^\pm = \frac{1}{2}(\sigma^1 \pm i\sigma^2)$. It is natural to first of all mod out by the reflection R, such that we have only one patch, labelled by $l = 0$. We can compute the metric on the group easily. The coordinates on this patch of the group are related to Poincare coordinates in a one-to-one fashion:

$$\begin{aligned} z &= e^{-v} \\ \lambda_+ &= \rho e^{-2v} \\ \lambda_- &= u \end{aligned} \tag{2.11}$$

where $z \in R^+$, $\lambda_\pm, \rho, u, v \in R$.

Checking carefully the coordinate transformations above, it becomes clear that the region $r > \sqrt{M/2}$ in the standard coordinates for the extremal BTZ black hole is mapped one-to-one to the half of the Poincare patch with $\lambda_+ > 0$. We can obtain that condition in the WZW model by modding out the model further by Bargmann's automorphism B, which allows us to choose the sign of λ_+.

In summary, the region outside the horizon of the extremal black hole is mapped one-to-one to the fundamental domain of the WZW model modded out by $(1, R, B, RB)$. Thus we have a global definition of the BTZ spacetime in terms of the group manifold. It can moreover be checked that the region inside the horizon can be included in the group theory description by not modding out by B. The extra patch can be mapped to the region inside the horizon using coordinate transformations very analogous to the ones above.

Of course we still need to mod out by the appropriate $\phi \to \phi + 2\pi$ symmetry to obtain the black hole spacetime. It would be useful to clarify the group theory description of the full extended extremal BTZ spacetime in [12], but we won't need it here.

2.4 Massless black hole

It turns out that the massless black hole case, with metric

$$ds^2 = -\frac{r^2}{l^2}dt^2 + \frac{l^2}{r^2}dr^2 + r^2 d\phi^2 \tag{2.12}$$

is the simplest to describe. We parametrise the group as:

$$g = e^{v_l L^-} e^{\rho\sigma^3} (i\sigma^2) e^{-v_r L^-} (-1)^l \, . \tag{2.13}$$

After taking the orbifold by the reflection R, and restricting to the fundamental domain labelled by $l = 0$, we cover the standard massless black hole background precisely once, and the range of the radial coordinate is $r = e^\rho \in \,]0, \infty[$. The boundary at infinity is clearly $R \times S^1$.

If we would not mod out by R, then the spacetime has a second component. We can interpret the resulting total group manifold as describing the massless black hole where the radial coordinate r runs over the real line $r \in \,]-\infty, \infty[$. In that case, we have to consider two boundaries of the form $R \times S^1$, one at each radial infinity. Note that the group generated by R is embedded in $\mathrm{SL}(2, \mathbb{R})$ as $Z_2 = \{1, -1\} \subset \mathrm{SL}(2, \mathbb{R})$. The representations of $\mathrm{SL}(2, \mathbb{R})$ are easily classified as having a definite parity under the Z_2. The above choice of whether to orbifold or not thus corresponds to the choice of including into the Hilbert space of the model representations with non-trivial parity or not.

Note that both in the case of the extremal and the massless black hole there is a group theoretic operation that maps the interior of the black hole to the exterior.

3. Massless black hole

After the analysis of some global aspects of the family of BTZ black holes, we can turn to the simplest cousin, the massless black hole. Some features of strings on the massless BTZ black hole background were uncovered already. In particular its connection to the

AdS_3/CFT_2 correspondence via spectral flow in the boundary conformal field theory [9, 10], the construction of the boundary conformal algebra in terms of worldsheet vertex operators [17], and the analysis of the ghost-free spectrum [24, 25] for winding strings [17] in the massless black hole background.

But simple aspects of the model were not explored very deeply. In particular, little has been said about the features of the model associated to the lorentzian nature of space-time. Although the analysis of the minisuperspace limit of $\mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$ (i.e. euclidean AdS_3) was performed in a well-controlled manner [26], as was the CFT analysis [27], little has been said about the appropriate lorentzian analytic continuation. (For discussions of the lorentzian AdS_3 in relation to the euclidean one see e.g. [23, 28, 29].) Although the lorentzian two-dimensional CFT is presumably unstable and may be difficult to make sense of without reference to its euclidean counterpart, one certainly can analyse lorentzian aspects of the AdS/CFT correspondence by simply studying a particle on the group manifold. It has been pointed out several times that it is important to get a better handle on time-dependent lorentzian features of the AdS_3/CFT_2 correspondence to make progress in understanding black hole physics via the corresponding boundary field theory.

By analysing the quantum mechanical particle on the $\mathrm{SL}(2,\mathbb{R})$ group manifold, in a way appropriate for the study of strings on the massless black hole background, we hope to provide good intuition for the correct lorentzian interpretation of the appropriate $\mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$ amplitudes.

3.1 Model

The metric for the vacuum black hole with $M = 0 = J$ is given by

$$ds^2 = -\frac{r^2}{l^2}dt^2 + \frac{l^2}{r^2}dr^2 + r^2 d\phi^2 \tag{3.1}$$

with $\phi \in [0, 2\pi[$ (and we put $l = 1$). In terms of the coordinates on the group we have (in the patch $l = 0$)

$$g = e^{v_l L^-} e^{\rho\sigma^3}(i\sigma_2)e^{-v_r L^-} \,. \tag{3.2}$$

The parametrisation is related to the previous coordinates by the transformation

$$\begin{aligned} v_l &= \phi + t \\ v_r &= \phi - t \\ e^\rho &= r \,. \end{aligned} \tag{3.3}$$

In terms of these coordinates the metric reads:

$$ds^2 = e^{2\rho} dv_l dv_r + d\rho^2 \,, \tag{3.4}$$

which is the invariant metric on the group manifold. The global angular identification in these coordinates is $(v_l, v_r, \rho) \equiv (v_l + 2\pi, v_r + 2\pi, \rho)$, which is generated by $(h_l, h_r) = (e^{2\pi L^-}, e^{-2\pi L^-})$.

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3.2 Minisuperspace

In this section we discuss the lagrangian, the hamiltonian and the symmetries of the particle model in some detail (see [30]). The lagrangian density in the minisuperspace limit is:

$$\mathcal{L} = \frac{1}{2}\left(\dot{\rho}^2 + e^{2\rho}\dot{v}_l\dot{v}_r\right) . \tag{3.5}$$

The variables canonically conjugate to the coordinates are:

$$\begin{aligned} p_l &= \frac{1}{2}e^{2\rho}\dot{v}_r \\ p_r &= \frac{1}{2}e^{2\rho}\dot{v}_l \\ p_\rho &= \dot{\rho} \, . \end{aligned} \tag{3.6}$$

The classical hamiltonian is:

$$\mathcal{H} = \frac{p_\rho^2}{2} + 2e^{-2\rho}p_l p_r = \frac{p^\mu p_\mu}{2} \, . \tag{3.7}$$

To quantise the model, we impose the commutation relations:

$$\begin{aligned} [\rho, p_\rho] &= i \\ [v_l, p_l] &= i \\ [v_r, p_r] &= i \, . \end{aligned} \tag{3.8}$$

The symmetry of the model is $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$, which will be broken to $R \times \mathrm{U}(1)$ (i.e. time translations and rotations) by the global angular identification. We can define the currents as $J_L^i = k\,Tr(L^i \partial_+ g g^{-1})$ for the leftmovers, and $J_R^i = -k\,Tr(L^i g^{-1}\partial_- g)$ for the rightmovers, where $L^3 = \frac{i}{2}\sigma^3$. The leftmoving currents can be computed to be:

$$\begin{aligned} J_L^3 &= ik(\partial_\tau \rho - e^{2\rho} v_l \partial_\tau v_r) \\ J_L^- &= k e^{2\rho}\partial_\tau v_r \\ J_L^+ &= k(\partial_\tau v_l + 2v_l\partial_\tau \rho - v_l^2 e^{2\rho}\partial_\tau v_r) \end{aligned} \tag{3.9}$$

and the rightmoving ones are:

$$\begin{aligned} J_R^3 &= ik(\partial_\tau \rho - e^{2\rho} v_r \partial_\tau v_l) \\ J_R^- &= k e^{2\rho}\partial_\tau v_l \\ J_R^+ &= k(\partial_\tau v_r + 2v_r\partial_- \rho - v_r^2 e^{2\rho}\partial_\tau v_l) \, . \end{aligned} \tag{3.10}$$

In terms of the canonical variables, we can rewrite the currents as:

$$\begin{aligned} J_L^- &= 2kp_l \\ J_L^3 &= ik(p_\rho - (v_l p_l + p_l v_l)) \\ J_L^+ &= 2k(e^{-2\rho}p_r + v_l p_\rho - v_l p_l v_l) \end{aligned} \tag{3.11}$$

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where we chose the ordering as to have (anti)-hermitian currents (and we could have added normal ordering constants). We have similar formulas for the right-movers. With these conventions, the commutation relations between the currents are:

$$\begin{aligned} \left[J_L^3, J_L^{\mp}\right] &= \pm 2k J_L^{\mp} \\ \left[J_L^-, J_L^+\right] &= -4k J_L^3 \,, \end{aligned} \tag{3.12}$$

which indeed generates an $sl(2,\mathbb{R})$ algebra. The quadratic Casimir c_2 is related to the quantum mechanical hamiltonian H for a particle on a group manifold. We have:

$$c_2 = -\frac{H}{2} - \frac{1}{4} \,. \tag{3.13}$$

3.3 Hilbert space

The first slightly trickier issue to tackle is the choice of Hilbert space for the particle on the group manifold. We propose to take the Hilbert space to consist of the quadratically integrable functions on the group manifold $\mathrm{SL}(2,\mathbb{R})$, and to mod it out by the global angular identification. In a string theoretic context, that gives rise to twisted states, namely the winding states discussed in detail in this context in [17] following e.g. [31, 32]. For the string spectrum for $\mathrm{SL}(2,\mathbb{R})$, see also [29].

The (scalar) particle wavefunction in a second quantisation satisfies the Klein-Gordon equation. We can classify the solutions using the two commuting conserved quantities, p_l and p_r, which correspond to the difference and the sum of the energy E and the angular momentum L. The Klein-Gordon equation in our curved background is:

$$\Delta\Phi(v_l, v_r, \rho) = \tau(\tau+1)\Phi \,, \tag{3.14}$$

where we used the standard form for the quadratic Casimir $c_2 = \tau(\tau+1)$. The equation has solutions:

$$\begin{aligned} \Phi &= e^{-i\lambda v_l} e^{i\mu v_r} u(\rho) \\ &= e^{-iEt} e^{iL\phi} u(\rho) \end{aligned} \tag{3.15}$$

where the energy E is given by $E = \lambda + \mu$ and the angular momentum by $L = \mu - \lambda$. The function of the radial variable, $u(\rho)$, can be expressed in terms of a linear combination of Bessel functions.

The case $E^2 > L^2$. For $4\lambda\mu = E^2 - L^2 > 0$, we find the solutions:

$$\Phi = e^{-iEt} e^{iL\phi} e^{-\rho} \left(c_1 J_{2\tau+1}\left(\sqrt{4\lambda\mu}\, e^{-\rho}\right) + c_2 J_{-(2\tau+1)}\left(\sqrt{4\lambda\mu}\, e^{-\rho}\right)\right) . \tag{3.16}$$

For integer values of the index, we can express the second linearly independent solution in terms of $Y_{2\tau+1}$.

The case $E^2 < L^2$. For $4\lambda\mu = E^2 - L^2 < 0$, we find the solutions:

$$\Phi = e^{-iEt}e^{iL\phi}e^{-\rho}\left(c_1 I_{2\tau+1}\left(\sqrt{-4\lambda\mu}e^{-\rho}\right) + c_2 I_{-(2\tau+1)}\left(\sqrt{-4\lambda\mu}e^{-\rho}\right)\right). \quad (3.17)$$

For integer values of the index, we need $K_{2\tau+1}$ to find a second linearly independent solution.[2]

3.4 Normalisability and unitarity

If we restrict to unitary representations of $\mathrm{SL}(2,\mathbb{R})$, the possible values for τ are $\tau \in \frac{1}{2}Z$ or $-1 < \tau < 0$ or $\tau = -1/2 + is$ where $s \in R$. When we further restrict our Hilbert space to the quadratically integrable functions,[3] we only need the principal continuous ($\tau = -1/2 + is$) and discrete ($\tau \in \frac{1}{2}Z$) representations.

It can be shown that the space of quadratically integrable functions can be decomposed into an integral over continuous representations (with two possible parities) and discrete representations with highest or lowest weight ([33] p. 488). For those representations, the behavior of the particle wavefunctions at infinity is like plane waves (continuous representations, delta-function normalisable) or exponential fall-off (discrete representations).[4]

We return to a more detailed description of this decomposition when discussing two-point functions in the minisuperspace limit.

3.5 Kernels

The wavefunctions for the scalar particle can also be expressed as matrix elements, or kernels, of the representations of $\mathrm{SL}(2,\mathbb{R})$. Indeed, the kernels K^{33} of $\mathrm{SL}(2,\mathbb{R})$ expressed in a parabolic basis (i.e. diagonalising p_r and p_l), are eigenfunctions of the laplacian with eigenvalue $\tau(\tau+1)$, and represent wavefunctions of scalar fields in a momentum basis. When normalized properly, they are given by [33]:

$$\begin{aligned} K^{33}(\lambda,\mu;\chi;g) &= (-1)^{2\epsilon\delta}\frac{e^{i\pi\epsilon}}{2\sin\left(\pi\left(\tau+\epsilon+\frac{1}{2}\right)\right)}\left(\frac{\mu}{\lambda}\right)^{\tau+\frac{1}{2}} e^{-i(\lambda+\mu)t}e^{i(\mu-\lambda)\phi}e^{-\rho}\times \\ &\quad\times\left(J_{-2\tau-1}\left(\sqrt{4\lambda\mu}e^{-\rho}\right) - (-1)^{2\epsilon}J_{2\tau+1}\left(\sqrt{4\lambda\mu}e^{-\rho}\right)\right), \end{aligned} \quad (3.18)$$

for $\lambda\mu > 0$. Here, $\epsilon = 0, 1/2$ denotes the parity of the representation; $\delta = \frac{1-\mathrm{sign}(\lambda)}{2}$; $\chi = (\tau,\epsilon)$ and g is given by (3.2). We have moreover:

$$\begin{aligned} K^{33}(\lambda,\mu;\chi;g) &= (-1)^{2\epsilon\delta}\frac{1}{\pi}\left(-\frac{\mu}{\lambda}\right)^{\tau+\frac{1}{2}} e^{-i(\lambda+\mu)t}e^{i(\mu-\lambda)\phi}e^{-\rho}\left(e^{(\tau+\frac{1}{2})\pi i} + (-1)^{2\epsilon}e^{-(\tau+\frac{1}{2})\pi i}\right)\times \\ &\quad\times K_{2\tau+1}\left(\sqrt{-4\lambda\mu}e^{-\rho}\right), \end{aligned} \quad (3.19)$$

for $\lambda\mu < 0$. If we include $(-1)^l$ in g, the kernels are multiplied by $(-1)^{2\epsilon l}$.

[2] Note that we don't treat the special case $E^2 = L^2$ here, although is has interesting special properties (see e.g. [17]) related to supersymmetry.

[3] By square integrable functions, we mean the functions square integrable against the group measure $dg = \frac{1}{4\pi^2}d\phi dt d\rho e^{2\rho}$, where we integrate ρ, t over the real line and ϕ becomes an angular variable after the compactification.

[4] There are two linearly independent solutions to the Klein-Gordon equation. For the discrete series, only one, which falls off exponentially, appears as a (normalisable) matrix element. The other solution, which exponentially increases, is crucial for AdS/CFT though, as it provides a non-fluctuating background, or non-trivial source term on the boundary [34].

These formulas are valid when $-1 < \mathrm{Re}(\tau) < 0$, but can easily be extended to the unitary discrete series. We thus find that the matrix elements of the unitary, square integrable representation spaces correspond to the particle wavefunctions that we computed before. The global identification $\phi \to \phi + 2\pi$ restricts the choice of angular momentum $L = \mu - \lambda$ to be integer. We now want to turn to computing overlaps of these wavefunctions.

Remark. Note that once we properly defined the Hilbert space for the particle, there is a natural guess for the string spectrum in this background. We take the particle Hilbert space and build affine representations on these primary states to obtain the untwisted sector of the string Hilbert space. Next, we include winding, twisted sectors along the lines proposed in [17] by analogy to the AdS_3 case [31].

In the massless black hole case, it turns out that such the twisted states are related to the Hilbert space proposed in [24, 25] (for the continuous representations). Similarly, the inclusion of the twisted sectors in the generic case [16] following [31] essentially leads to the construction presented in [14, 35]. It would be nice to confirm the prediction for the Hilbert space by computing the one-loop partition function along the lines of [32].

4. Overlaps

In the previous section we constructed the particle Hilbert space using results from $\mathrm{SL}(2,\mathbb{R})$ group theory. In this section, we want to compute overlaps of the constructed wavefunctions. These are particle approximations (see e.g. [26]) to N-point functions in the CFT that one obtains when putting a string in the massless black hole background. For convenience we work in momentum space in this section, since that is where we most easily obtained the wavefunctions.

In our analysis of the two-point functions, we will clarify further the structure of the Hilbert space. When we come to the three-point functions, we will lay bare their $\mathrm{SL}(2,\mathbb{R})$ group structure.

We define the N-point function in momentum space in the minisuperspace limit to be given by the N-point overlap:

$$\begin{aligned}\left\langle \prod_{i=1}^{N} K^{33}(\lambda_i,\mu_i;\chi_i;g)\right\rangle &= \frac{1}{8\pi^2}\int dv_r \int dv_l \int_{-\infty}^{+\infty} d\rho e^{2\rho} \prod_{i=1}^{N} K^{33}(\lambda_i,\mu_i;\chi_i;g) \qquad (4.1)\\ &= \delta\left(\Sigma(\mu_i+\lambda_i)\right)\delta_{\Sigma(\mu_i-\lambda_i)} \int_{-\infty}^{+\infty} d\rho\, e^{2\rho} \prod_{i=1}^{N} K^{33}(\lambda_i,\mu_i;\chi_i;\rho)\,.\end{aligned}$$

This amounts to imposing conservation of energy and angular momentum, and evaluating an integral over N Bessel functions of argument $a_i e^{-\rho}$ with measure $e^{(2-N)\rho}\, d\rho$, where $a_i = \sqrt{\pm\lambda_i\mu_i}$. For convenience, we have made a summary of some of the integrals of Bessel functions that we need in calculating two- and three-point functions in our appendix A.

4.1 Two-point function

In this subsection, we derive the two-point overlaps for the kernels above. Of course, these are diagonal in the representation index, but since the proof of this fact entails some non-trivial functional analysis, we believe it is useful to incorporate it here. We first discuss the orthogonality of the Bessel-functions, and then discuss how the decomposition of our Hilbert space fits into a generalized theory of Fourier transforms. Then the analysis of the two-point functions becomes trivial.

Orthogonality of J_μ. We believe it is interesting to first of all show how the orthogonality relations among Bessel functions J_μ can be derived from the integral formula

$$(J_\mu, J_\nu) \equiv \int_0^\infty \frac{dt}{t} J_\mu(at) J_\nu(at) = \frac{2}{\pi} \frac{1}{\mu^2 - \nu^2} \sin \frac{\pi}{2}(\mu - \nu) \,, \tag{4.2}$$

which is valid for

$$\mathrm{Re}(\mu + \nu) > 0 \,, \qquad a > 0 \,. \tag{4.3}$$

Detailed example. Let us first consider the orthogonality of $J_{i\rho} \pm J_{-i\rho}$. To maintain (4.3), we regularlise it as $J_{i\rho+\epsilon} \pm J_{-i\rho+\epsilon}$ with a positive infinitesimal number ϵ. Using

$$(J_{i\rho+\epsilon}, J_{i\lambda+\epsilon}) = \frac{2i}{\pi} \frac{\sinh \frac{\pi}{2}(\rho - \lambda)}{\rho - \lambda} \frac{-1}{\rho + \lambda - 2i\epsilon} \,, \tag{4.4}$$

and similar expressions, one finds that

$$\begin{aligned}
(J_{i\rho+\epsilon} \pm J_{-i\rho+\epsilon}, J_{i\lambda+\epsilon} \pm J_{-i\lambda+\epsilon}) &= (J_{i\rho+\epsilon}, J_{i\lambda+\epsilon}) + (J_{-i\rho+\epsilon}, J_{-i\lambda+\epsilon}) \pm \\
&\quad \pm \left[(J_{i\rho+\epsilon}, J_{-i\lambda+\epsilon}) + (J_{-i\rho+\epsilon}, J_{i\lambda+\epsilon}) \right] \\
&= \frac{4}{\pi} \frac{\sinh \frac{\pi}{2}(\rho - \lambda)}{\rho - \lambda} \frac{2\epsilon}{(\rho + \lambda)^2 + (2\epsilon)^2} \pm \\
&\quad \pm \frac{4}{\pi} \frac{\sinh \frac{\pi}{2}(\rho + \lambda)}{\rho + \lambda} \frac{2\epsilon}{(\rho - \lambda)^2 + (2\epsilon)^2} \to \\
&\quad \to 2 \frac{\sinh \pi\rho}{\rho} \left[\delta(\rho + \lambda) \pm \delta(\rho - \lambda) \right] .
\end{aligned} \tag{4.5}$$

Here we used the relation $\delta(x) = \lim_{\epsilon \to +0} \frac{1}{\pi} \frac{\epsilon}{x^2 + \epsilon^2}$.

Other cases. Similarly, we find the following orthogonality relations:

$$\begin{aligned}
(J_{i\rho} + J_{-i\rho}, J_{2n}) &= 0 \\
(J_{i\rho} - J_{-i\rho}, J_{2n+1}) &= 0 \,.
\end{aligned} \tag{4.6}$$

and

$$\begin{aligned}
(J_{2n}, J_{2m}) &= \frac{1}{4n} \delta_{n,m} \,, \\
(J_{2n+1}, J_{2m+1}) &= \frac{1}{4n + 2} \delta_{n,m} \,.
\end{aligned} \tag{4.7}$$

Expansions of functions. The above calculation shows that the sets $\{J_{i\rho}+J_{-i\rho}, J_{2n}\,|\,\rho \in R_{\geq 0},\, n \in Z_{\geq 1}\}$, and $\{J_{i\rho} - J_{-i\rho}, J_{2n+1}\,|\,\rho \in R_{\geq 0},\, n \in Z_{\geq 0}\}$ form orthogonal sets of functions, respectively. Actually, in [36] it is shown, using the general theory of eigenfunctions of second order differential operators, that they form a complete set such that we can expand every quadratically integrable function on the half-line in terms of these orthogonal bases. From the expression for the kernels (3.18), it is clear that the two sets are related to trivial and non-trivial representations of the reflection symmetry R that we discussed in subsection 2.4. Thus, when we restrict to studying the unextended massless BTZ black hole, we would take along only wavefunctions with trivial parity $\epsilon = 0$. When we include the second patch in the group manifold (associated to $l = 1$), we need to add a second component to our Hilbert space, associated to wavefunctions with non-trivial parity $\epsilon = 1$.

We can expand a function $g(z)$ on the half-line as (see also [36])

$$\begin{aligned} g(z) &= \sum_{n=1}^{\infty} 4nC_n J_{2n}(z) + \int_0^\infty \frac{\rho d\rho}{2\sinh\pi\rho} C_\rho \{J_{i\rho}(z) + J_{-i\rho}(z)\}\,, \\ |g(z)|^2 &= (g, g^*) = \sum_{n=1}^{\infty} 4n|C_n|^2 + \int_0^\infty \frac{\rho d\rho}{2\sinh\pi\rho}|C_\rho|^2\,, \end{aligned} \tag{4.8}$$

where

$$C_n = (g, J_{2n})\,, \qquad C_\rho = (g, J_{i\rho} + J_{-i\rho})\,. \tag{4.9}$$

For the expansion by the second set of orthonormal functions we have:

$$\begin{aligned} g(z) &= \sum_{n=0}^{\infty} (4n+2) C_n J_{2n+1}(z) - \int_0^\infty \frac{\rho d\rho}{2\sinh\pi\rho} C_\rho \{J_{i\rho}(z) - J_{-i\rho}(z)\}\,, \\ |g(z)|^2 &= (g, g^*) = \sum_{n=0}^{\infty} (4n+2)|C_n|^2 + \int_0^\infty \frac{\rho d\rho}{2\sinh\pi\rho}|C_\rho|^2\,, \end{aligned} \tag{4.10}$$

where

$$C_n = (g, J_{2n+1})\,, \qquad C_\rho = (g, J_{i\rho} - J_{-i\rho})\,. \tag{4.11}$$

Orthogonality of K_μ. Similarly to the case of J_μ, we can derive the orthogonality of K_μ from the formula in the appendix, yielding:

$$(K_{i\rho}, K_{i\lambda}) \to \frac{\pi^2}{2\rho\sinh\pi\rho}[\delta(\rho+\lambda) + \delta(\rho-\lambda)]\,. \tag{4.12}$$

The function space can be decomposed as follows:

$$\begin{aligned} g(t) &= \frac{2}{\pi^2}\int_0^\infty C_\rho K_{i\rho}(t) \rho \sinh\pi\rho d\rho\,, \\ |g(t)|^2 &= (g, g^*) = \frac{2}{\pi^2}\int_0^\infty |C_\rho|^2 \rho\sinh\pi\rho d\rho\,, \end{aligned} \tag{4.13}$$

where

$$C_\rho = (g, K_{i\rho})\,. \tag{4.14}$$

This again agrees with [36]. These spaces of functions describe the components of the Hilbert space associated with particles that have energy larger or smaller than their angular momentum respectively. The double overlaps follow trivially from the above results.

4.2 Three-point function

We have clarified the structure of the Hilbert space, and we computed the double overlaps in the previous section. The three-point functions in our minisuperspace limit in lorentzian signature can be computed using known integrals of the product of three Bessel functions. We discuss the case for a triple overlap between wavefunctions in the continuous series in some detail. The other cases are interesting, and analogous. We then move on to make the connection to the Clebsch-Gordan coefficients of $\mathrm{SL}(2,\mathbb{R})$ in a parabolic basis.

Correlation functions for the continuous series. The overlaps for the principal continuous series with $2\tau_i + 1 = \nu_i \in iR$ are given by

$$\left\langle \prod_{i=1}^{N} K^{33}(\lambda_i, \mu_i; g; \chi_i) \right\rangle = \frac{1}{8\pi^2} \int dv_r \int dv_l \int_{-\infty}^{+\infty} d\rho\, e^{2\rho} \prod_{i=1}^{N} K^{33}(\lambda_i, \mu_i; g; \chi_i) \\ = \delta(\Sigma(\mu_i + \lambda_i)) \delta_{\Sigma(\mu_i - \lambda_i)} \int_{-\infty}^{+\infty} d\rho\, e^{2\rho} \prod_{i=1}^{N} K^{33}(\lambda_i, \mu_i; \rho; \chi_i)\,. \tag{4.15}$$

Since K^{33} are functions of $x = e^{-\rho}$, we find that calculation of the N-point functions reduces to integrals of the form

$$\int_0^\infty dx x^{N-3} \prod_{k=1}^{N} Z_{\pm\nu_i}(a_i x)\,, \tag{4.16}$$

where $Z_{\nu_i} = J_{\nu_i}$ or K_{ν_i} ($\nu_i \in iR$, $a_i > 0$). Let's concentrate on the three-point functions. To proceed, we need the integrals (A.7)–(A.10) in the appendix, with $\rho = 1$. Since $\nu_i \in iR$, $a_i > 0$ in (4.16), we find that the conditions for the formulas (A.11) are always satisfied, except for $a_3 > a_1 + a_2$ for (A.7). However, we can choose a_i so that $a_3 \geq a_1 + a_2$ is satisfied. From the conservation of λ_i, μ_i, we have

$$\lambda_1 + \lambda_2 + \lambda_3 = 0\,, \mu_1 + \mu_1 + \mu_3 = 0\,. \tag{4.17}$$

Note that (A.7) is needed only when $\lambda_i \mu_i > 0$, and denote $a_i = 2\sqrt{\lambda_i \mu_i}$. Then we obtain:

$$a_3^2 - (a_1 + a_2)^2 = 4\left(\sqrt{\lambda_1 \mu_2} - \sqrt{\lambda_2 \mu_1}\right)^2\,. \tag{4.18}$$

Also, from (4.17), two of $\lambda_i(\mu_i)$ have the same sign. So, we choose $\lambda_{1,2}(\mu_{1,2})$ so that they have the same sign and $a_3 \geq a_1 + a_2$. Moreover, we can set one of $\lambda_1 \mu_2$ and $\lambda_2 \mu_1$ is greater than or equal to the other. Thus, in the following we assume

$$a_3 > a_1 + a_2\,, \lambda_1 \mu_2 \geq \lambda_2 \mu_1 \tag{4.19}$$

for (A.12), without loss of generality.

Three-point function of K^{33} with $\lambda_a\mu_a > 0$. We will compute the triple overlap of two kernels with a third, complex conjugated kernel. We will schematically denote the overlap $\langle KKK^*\rangle$ below. The signs of λ_3 and μ_3 are flipped in (4.17) under complex conjugation. It will be convenient in this case to make an even more restrictive choice for the angular momentum and energy. We take all $\lambda_i, \mu_i > 0$. All other cases can be treated analogously, although we don't exhibit them in detail. Under these conditions, $z_\pm$ in (A.12) become

$$z_+ = \frac{\lambda_1}{\lambda_3}, \qquad z_- = \frac{\mu_2}{\mu_3}. \tag{4.20}$$

We have that the parameter $\delta_i = 0$ in the kernels. Here we first consider the overlap for $\mathrm{SL}(2,\mathbb{R})$ to facilitate the connection to the group theory discussed in the next subsection. This leads to summing over the two patches with $l = 0$ and $l = 1$, and to the parity conservation law $2\sum\epsilon_i \equiv 0 \,\mathrm{mod} 2$. We then obtain:

$$\begin{aligned}\langle KKK^*\rangle &= \frac{i\delta(\lambda_1+\lambda_2-\lambda_3)\delta(\mu_1+\mu_2-\mu_3)}{2^3\sin\pi(\frac{\nu_1}{2}+\epsilon_1)\sin\pi(\frac{\nu_2}{2}+\epsilon_2)\sin\pi(\frac{-\nu_3}{2}+\epsilon_3)}\times \\ &\quad\times\left(\frac{\mu_1}{\lambda_1}\right)^{\frac{\nu_1}{2}}\left(\frac{\mu_2}{\lambda_2}\right)^{\frac{\nu_2}{2}}\left(\frac{\lambda_3}{\mu_3}\right)^{\frac{\nu_3}{2}}\sum_{\sigma_i=\pm1}(-1)^{\Sigma\sigma_i(\epsilon_i+\frac{1}{2})}I_3(\sigma_i\nu_i)\,,\end{aligned} \tag{4.21}$$

where I_3 denotes the integral of three Bessel functions $J_{\sigma_i\nu_i}(a_i x)$ given in the appendix in formula (A.12). We will reshape this result for the three-point overlap such that it will become clearer how it relates to the Clebsch-Gordan coefficients of $\mathrm{SL}(2,\mathbb{R})$ in a parabolic basis. First of all, we remark that it is easy to sum the two contributions arising from $\sigma_3 = \pm1$. Next, we use the formula

$${}_2F_1(a,b;c;z) = (1-z)^{-a}\,{}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right), \tag{4.22}$$

to express the hypergeometric functions in terms of the arguments $-\lambda_1/\lambda_2$ and $-\mu_2/\mu_1$. Thus, we find the following slightly more compact expression for the three-point function:

$$\begin{aligned}\langle KKK^*\rangle &= \frac{i(-1)^{\epsilon_3+\frac{1}{2}}\delta(\lambda_1+\lambda_2-\lambda_3)\delta(\mu_1+\mu_2-\mu_3)}{8\pi\sin\pi(\frac{\nu_1}{2}+\epsilon_1)\sin\pi(\frac{\nu_2}{2}+\epsilon_2)}\lambda_2^{\frac{-1-\nu_1-\nu_2+\nu_3}{2}}\mu_1^{\frac{-1+\nu_1-\nu_2-\nu_3}{2}}\mu_2^{\nu_2}\times \\ &\quad\times\sum_{\sigma_{1,2}=\pm1}(-1)^{\sigma_1(\epsilon_1+\frac{1}{2})+\sigma_2(\epsilon_2+\frac{1}{2})}\left(\frac{\lambda_2}{\lambda_1}\right)^{\nu_1(1-\sigma_1)/2}\left(\frac{\mu_1}{\mu_2}\right)^{\nu_2(1-\sigma_2)/2}\times \\ &\quad\times\sin\frac{\pi}{2}(-\sigma_1\nu_1-\sigma_2\nu_2+2\epsilon_3)\Gamma\begin{bmatrix}\frac{1+\sigma_1\nu_1+\sigma_2\nu_2+\nu_3}{2}, \frac{1+\sigma_1\nu_1+\sigma_2\nu_2-\nu_3}{2} \\ \sigma_1\nu_1+1, \sigma_2\nu_2+1\end{bmatrix}\times \\ &\quad\times{}_2F_1\left(\frac{1+\sigma_1\nu_1+\sigma_2\nu_2-\nu_3}{2}, \frac{1+\sigma_1\nu_1-\sigma_2\nu_2-\nu_3}{2}; \sigma_1\nu_1+1; -\frac{\lambda_1}{\lambda_2}\right)\times \\ &\quad\times{}_2F_1\left(\frac{1+\sigma_1\nu_1+\sigma_2\nu_2+\nu_3}{2}, \frac{1-\sigma_1\nu_1+\sigma_2\nu_2+\nu_3}{2}; \sigma_2\nu_2+1; -\frac{\mu_2}{\mu_1}\right),\end{aligned} \tag{4.23}$$

with $\Gamma\left[\begin{smallmatrix}a_1,a_2,\dots\\ b_1,b_2,\dots\end{smallmatrix}\right] = \frac{\Gamma(a_1)\Gamma(a_2)\dots}{\Gamma(b_1)\Gamma(b_2)\dots}$. This is a rather complicated formula for the three-point function in the momentum basis. We will gain a lot of insight in it by rewriting it as a product of Clebsch-Gordan coefficients. We have moulded it into a form that will naturally arise from the group theory analysis in the next section. For the unextended massless black hole, we have only to set $\epsilon_i = 0$ and replace the delta functions as in (4.15).

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4.3 Underlying group theory

These three-point overlaps are related to the Clebsch-Gordan coefficients of $\mathrm{SL}(2,\mathbb{R})$ in a parabolic basis. Unfortunately, these CGC's are not known in a parabolic basis, so we need to derive them. That is a non-trivial exercise in group theory. Results on CGC's in an elliptic basis, relevant for strings in AdS_3, were known (see e.g. [37] and references therein). We follow some of the techniques developed in that context to derive the CGC's in a parabolic basis. In particular, we follow closely the analysis in [38]. We start by considering homogeneous functions that carry an $\mathrm{SL}(2,\mathbb{R})$ representation:

$$(T_\chi(g)F)\,(x,y) = F(\alpha x + \gamma y, \beta x + \delta y)\,. \tag{4.24}$$

The function F satisfies:

$$F(cx,cy) = |c|^{2\tau}\,\mathrm{sign}^{2\epsilon}(c) F(x,y)\,, \tag{4.25}$$

and we will characterize the representation by its quadratic Casimir and parity $\chi = (\tau,\epsilon)$. We can take the same representation to be defined on a function f on the line $(x,1)$ parametrized by x. (We will often denote the one argument of f by x, abusing notation. We hope that does not lead to confusion with the first argument of $F(x,y)$.) The representation acts as:

$$(T_\chi(g)f)(x) \;=\; |\beta x+\delta|^{2\tau}\,\mathrm{sign}^{2\epsilon}(\beta x+\delta) f\left(\frac{\alpha x+\gamma}{\beta x+\delta}\right). \tag{4.26}$$

We can define a kernel K that intertwines the product of two representations with a third:

$$f(x_1;\chi_1)f(x_2;\chi_2) \;=\; \int K^{\chi_3}_{\chi_1\chi_2}(x_1,x_2;x_3) f(x_3;\chi_3)dx_3dj_3\,. \tag{4.27}$$

The integral is over the space of irreducible representations (labelled by j_3), and the quantum numbers in those representations (labelled by x_3). By acting on both sides of the equation with $T(g)$, we can find a relation between the kernel and its transform. We obtain:

$$\begin{aligned} K(x_i) \;=\; & K\left(\frac{\alpha x_i+\gamma}{\beta x_i+\delta}\right)|\beta x_1+\delta|^{2\tau_1}\,\mathrm{sign}^{2\epsilon_1}(\beta x_1+\delta)|\beta x_2+\delta|^{2\tau_2}\,\mathrm{sign}^{2\epsilon_2}(\beta x_2+\delta)\times \\ & \times|\beta x_3+\delta|^{-2\tau_3-2}\,\mathrm{sign}^{2\epsilon_3}(\beta x_3+\delta)\,. \end{aligned} \tag{4.28}$$

Now we look for a solution to this equation of the form suggested by the $\mathrm{SL}(2,\mathbb{R})$ invariant $(x_i y_j - y_i x_j)$:

$$\begin{aligned} K(x_i) \;=\; & \kappa|x_1-x_2|^{-2a_3}|x_2-x_3|^{-2a_1}|x_3-x_1|^{-2a_2}\times \\ & \times\,\mathrm{sign}^{2\eta_3}(x_1-x_2)\,\mathrm{sign}^{2\eta_1}(x_2-x_3)\,\mathrm{sign}^{2\eta_2}(x_3-x_1)\,, \end{aligned} \tag{4.29}$$

with $\eta_i = 0, 1/2$. We then find two solutions of this form parametrised by $\omega \equiv \sum \eta_i \bmod 1$.

In the light of the representation theory of $SL(2,\mathbb{R})$ it is no surprise that we find two solutions, since the product of two continuous representations contains two copies of each irreducible continuous representation (e.g. [33]). We also have $\sum 2\epsilon_i \equiv 0 \bmod 2$ and

$$\begin{aligned} 2a_1 &= \tau_1 - \tau_2 + \tau_3 + 1 \\ 2a_2 &= -\tau_1 + \tau_2 + \tau_3 + 1 \\ 2a_3 &= -\tau_1 - \tau_2 - \tau_3 - 1 \\ 2\eta_i &\equiv 2\epsilon_i + 2\omega (\mathrm{mod} 2)\,. \end{aligned} \tag{4.30}$$

By expressing the relation (4.27) in an orthonormal basis of functions, we can show that the parabolic CGC are related through an overall constant to the following integral (see [33] for an alternative derivation):

$$\begin{aligned} C_\omega(\lambda) = \kappa \int_{-\infty}^{+\infty} & dx_1 dx_2 dx_3 |x_2 - x_3|^{-2a_1} \operatorname{sign}^{2\eta_1}(x_2 - x_3) |x_3 - x_1|^{-2a_2} \operatorname{sign}^{2\eta_2}(x_3 - x_1) \\ & |x_1 - x_2|^{-2a_3} \operatorname{sign}^{2\eta_3}(x_1 - x_2) e^{i(\lambda_1 x_1 + \lambda_2 x_2 - \lambda_3 x_3)}\,. \end{aligned} \tag{4.31}$$

In the appendix it is shown that these Clebsch-Gordan coefficients can be computed to be (for $\lambda_i > 0$):

$$\begin{aligned} C_\omega(\lambda) = & 8\kappa\delta(\lambda_1 + \lambda_2 - \lambda_3)\Gamma\left(\frac{1 - \nu_1 + \nu_2 - \nu_3}{2}\right)\Gamma\left(\frac{1 + \nu_1 - \nu_2 - \nu_3}{2}\right) \times \\ & \times \Gamma\left(\frac{1 + \nu_1 + \nu_2 + \nu_3}{2}\right) \prod_i \sin(\pi(a_i - \eta_i))(-i)^{2\eta_i} \lambda_2^{\frac{-\nu_1 - \nu_2 + \nu_3 - 1}{2}} \times \\ & \times \Bigg\{ {}_2F_1\left(\frac{\nu_1 - \nu_2 - \nu_3 + 1}{2}, \frac{\nu_1 + \nu_2 - \nu_3 + 1}{2}; \nu_1 + 1; -\frac{\lambda_1}{\lambda_2}\right) \times \\ & \times \Gamma\begin{bmatrix} \frac{\nu_1 - \nu_2 + \nu_3 + 1}{2}, \frac{\nu_1 + \nu_2 - \nu_3 + 1}{2} \\ \nu_1 + 1 \end{bmatrix} \times \\ & \times \left((-1)^{2\omega} - \frac{(-1)^{2\epsilon_1} \cos\frac{\pi}{2}(\nu_1 + \nu_2 - \nu_3) + \cos\frac{\pi}{2}(\nu_1 - \nu_2 + \nu_3)}{\sin \pi\nu_1} \right) + \\ & + \left(\frac{\lambda_1}{\lambda_2}\right)^{-\nu_1} {}_2F_1\left(\frac{-\nu_1 + \nu_2 - \nu_3 + 1}{2}, \frac{-\nu_1 - \nu_2 - \nu_3 + 1}{2}; -\nu_1 + 1; -\frac{\lambda_1}{\lambda_2}\right) \times \\ & \times \Gamma\begin{bmatrix} \frac{-\nu_1 + \nu_2 + \nu_3 + 1}{2}, \frac{-\nu_1 - \nu_2 - \nu_3 + 1}{2} \\ -\nu_1 + 1 \end{bmatrix} \times \\ & \times \left((-1)^{2\omega + 2\epsilon_2} + \frac{(-1)^{2\epsilon_1} \cos\frac{\pi}{2}(-\nu_1 + \nu_2 + \nu_3) + \cos\frac{\pi}{2}(\nu_1 + \nu_2 + \nu_3)}{\sin \pi\nu_1} \right) \Bigg\}\,, \end{aligned} \tag{4.32}$$

4.4 Towards the three-point function

To obtain the three-point function for continuous representations, we have taken into account the fact that there are two Clebsch-Gordan coefficients associated to each triple of continuous representations. These two CGC's were labelled by ω. We can compute now the following combination of the CGC's which does not depend on a choice of basis in each of these equivalent irreducible components in the tensor product representation:

$$\begin{aligned}
&C_{\omega=0}(\lambda)C_{\omega=0}(\mu)^* + C_{\omega=1/2}(\lambda)C_{\omega=1/2}(\mu)^* = \\
&\quad = |\kappa|^2(2\pi)^4 \frac{\delta(\lambda_1+\lambda_2-\lambda_3)\delta(\mu_1+\mu_2-\mu_3)}{\sin\pi(\frac{\nu_1}{2}+\epsilon_1)\sin\pi(\frac{\nu_2}{2}+\epsilon_2)} \lambda_2^{\frac{-1-\nu_1-\nu_2+\nu_3}{2}} \mu_1^{\frac{-1+\nu_1-\nu_2-\nu_3}{2}} \mu_2^{\nu_2} \times \\
&\qquad \times \sum_{\sigma_{1,2}=\pm1} (-1)^{\frac{1}{2}(\sigma_1+\sigma_2)} \left(\frac{\lambda_2}{\lambda_1}\right)^{\nu_1(1-\sigma_1)/2} \left(\frac{\mu_1}{\mu_2}\right)^{\nu_2(1-\sigma_2)/2} \times \\
&\qquad \times \sin\frac{\pi}{2}[\sigma_1(\nu_1+2\epsilon_1)+\sigma_2(\nu_2+2\epsilon_2)] \times \\
&\qquad \times \Gamma\left[\frac{1+\sigma_1\nu_1+\sigma_2\nu_2+\nu_3}{2}, \frac{1+\sigma_1\nu_1+\sigma_2\nu_2-\nu_3}{2} \right]\sigma_1\nu_1+1, \sigma_2\nu_2+1 \times \\
&\qquad \times {}_2F_1\left(\frac{1+\sigma_1\nu_1+\sigma_2\nu_2-\nu_3}{2}, \frac{1+\sigma_1\nu_1-\sigma_2\nu_2-\nu_3}{2}; \sigma_1\nu_1+1; -\frac{\lambda_1}{\lambda_2}\right) \times \\
&\qquad \times {}_2F_1\left(\frac{1+\sigma_1\nu_1+\sigma_2\nu_2+\nu_3}{2}, \frac{1-\sigma_1\nu_1+\sigma_2\nu_2+\nu_3}{2}; \sigma_2\nu_2+1; -\frac{\mu_2}{\mu_1}\right). \qquad (4.33)
\end{aligned}$$

To calculate $C_\omega(\mu)^*$, we first of all made use of the symmetry of the integral I' that we noted in the appendix, to express the CGC as a function of the argument $-\mu_2/\mu_1$ (making use of the fact that the μ_i are positive by choice). To that end we need to exchange ν_1 with ν_2, and $\lambda_{1,2}$ with $\mu_{2,1}$, and multiply with an overall $(-1)^{2\omega}$. Finally, to take the complex conjugate, we simply map ν_i to $-\nu_i$ for these continuous representations. After using (a lot of) standard trigonometric identities and comparing the coefficients of the hypergeometric functions, the result we find agrees with the computation of the triple overlap via the integral over three Bessel functions, for the choice of normalisation constant $|\kappa|^2 = \frac{1}{(2\pi)^3}\frac{1}{(4\pi)^2}$.

Of course, the three-point functions for other choices of representation, and energy and angular momentum can similarly be related to Clebsch-Gordan coefficients. It would be interesting to explicitly prove that statement, especially in the cases that involve discrete representations. The analysis should be doable using our results and the example of known results in the elliptic basis.

4.5 Four-point function

For the four-point overlaps, we need the integral of the product of four Bessel functions. These (along with integrals relevant for higher point-functions) can be found in [39]. We won't discuss the four-point function in detail here — we only make a few comments. The Fourier-transformed integral has been studied in the context of the AdS/CFT correspondence. Even before making that statement clearer in the next section, we already note that the analysis of the singular parts of the four-point function in the case of AdS_5/CFT_4 (see e.g. [40] for a review) gave rise to not only power law singularities, but also logarithmic singularities that were interpreted [41] as arising from operators with anomalous dimensions. It would certainly be interesting to revisit the analysis of the singularity structure in the two-dimensional context, to analyze factorisation of the four-point function (even at the level of the minisuperspace limit). Of course, one would keep an eye out for the singularities appearing in the four-point functions in ordinary (flat space) parabolic orbifolds [6], and analyze whether these arises in the context of the massless BTZ black hole. It would be interesting to see whether and how the instability mechanism pointed out in [8] is at work here.

5. Holographic dual

In this section, we want to show in what way our work is related to previous work on the AdS/CFT correspondence. One of the main differences is that we worked in lorentzian signature. Aspects of the correspondence in lorentzian signature have been studied in [42]–[44]. In particular the authors of [44] point out that it is natural to concentrate on Poincare patches within AdS space to formulate a lorentzian version of the correspondence. We note that in the context of the massless BTZ black hole, this is very naturally realised (see subsection 2.4).

In this section we review some holographic aspects of strings on BTZ black holes, and we add a few comments on aspects that are particular to the extremal and massless black hole. We believe this clarifies some open ends in the literature, and shows that our results can find applications in a more general context.

5.1 Bulk-to-boundary

It is well known in the euclidean AdS context that the bulk-boundary propagator in Poincaré coordinates is:

$$K_P(z, \lambda_+, \lambda_-) = c \left(\frac{z}{z^2 + \Delta\lambda_+ \Delta\lambda_-} \right)^{2h_+}, \tag{5.1}$$

where $c = \frac{2h_+ - 1}{\pi}$ and $2h_+$ is related to the scaling dimension of the boundary field. The bulk field Φ can then be written in terms of the boundary condition Φ_0:

$$\Phi(z, \lambda_+, \lambda_-) = \int d\lambda'_+ d\lambda'_- K_P(z, \lambda_+, \lambda_-; \lambda'_+, \lambda'_-) \Phi_0(\lambda'_+, \lambda'_-) \,. \tag{5.2}$$

In lorentzian signature, there are normalisable modes in the bulk that allow for many bulk solutions to the same boundary conditions. We assume, until subsection 5.4, that these take the values that follow from analytic continuation from the euclidean case, for simplicity. We thus specify a particular coherent state in the boundary conformal field theory [43].

Of course, we know that the Poincare coordinates are very appropriate for a description of the massless BTZ black hole, and we therefore needn't look further for the bulk-to-boundary propagator for the massless BTZ black hole. Let us nevertheless follow a roundabout route to reobtain the propagator to illustrate some important points.

We can derive the bulk-to-boundary propagator for the generic BTZ black hole background by performing some coordinate transformations starting from a Poincare patch. We need to carefully take into account the transformation of the measure in (5.2) as well as the conformal transformation of the boundary field (with weights $h_- = 1 - h_+$) (see e.g. [45]). The coordinate transformations to obtain the generic BTZ black hole metric are:

$$\begin{aligned}\lambda_\pm &= \sqrt{\frac{r^2-r_+^2}{r^2-r_-^2}}e^{2\pi T_\pm u_\pm}\\ z &= \sqrt{\frac{r_+^2-r_-^2}{r^2-r_-^2}}e^{\pi(T_+u_++T_-u_-)},\end{aligned} \tag{5.3}$$

where $u_\pm = -\phi \pm t$. A computation (with ϕ non-compact) yields the propagator:[5]

$$\begin{aligned}K_{BTZ} &= c(4\pi^2 T_+T_-)^{2h_+}\left(\frac{r^2-r_-^2}{(r^2-r_+^2)^2}\right)^{h_+} e^{-2\pi h_+(T_+\Delta u_++T_-\Delta u_-)}\times\\ &\times\left(\frac{r_+^2-r_-^2}{r^2-r_+^2}+\left(1-e^{-2\pi T_+\Delta u_+}\right)\left(1-e^{-2\pi T_-\Delta u_-}\right)\right)^{-2h_+}.\end{aligned} \tag{5.4}$$

Note that we did not make the approximation used in [45], which explains for the slightly different expression for the propagator. Similarly, following the coordinate transformations in [10] or our section 2, we obtain the bulk-boundary propagators for the extremal and the massless BTZ black hole (for ϕ non-compact):

$$\begin{aligned}K_{EBH} &= ce^{-2\pi h_+T_+\Delta u_+}\left(\frac{1}{r^2-r_+^2}\right)^{h_+}\left(\frac{1}{r^2-r_+^2}+\frac{1}{2\pi T_+}\left(1-e^{-2\pi T_+\Delta u_+}\right)\Delta u_-\right)^{-2h_+}\\ K_{M=0} &= c\left(\frac{r}{1+r^2\Delta u_+\Delta u_-}\right)^{2h_+}.\end{aligned} \tag{5.5}$$

It is a posteriori obvious that, although the coordinate transformation (5.3) to go from the Poincare patch to the generic BTZ background degenerates for the extremal cases, the bulk-to-boundary propagator for these extremal cases can be obtained straightforwardly by taking the appropriate limit of the bulk-to-boundary propagator. Thus, we needn't be worried when taking limits of the generic BTZ results based on the bulk-to-boundary propagator.

As we anticipated, the bulk-to-boundary propagator that we obtained for the massless black hole is the Poincare bulk-to-boundary propagator.

Finally, to obtain the propagators taking into account the periodicity of ϕ, we need to sum the propagator over images [19]:

$$\begin{aligned}K_{BTZ} &= \sum_{n=\infty}^{-\infty} c(4\pi^2T_+T_-)^{2h_+}\left(\frac{r^2-r_-^2}{(r^2-r_+^2)^2}\right)^{h_+} e^{-2\pi h_+(T_+(\Delta u_++2\pi n)+T_-(\Delta u_-+2\pi n))}\times\\ &\times\left(\frac{r_+^2-r_-^2}{r^2-r_+^2}+\left(1-e^{-2\pi T_+(\Delta u_++2\pi n)}\right)\left(1-e^{-2\pi T_-(\Delta u_-+2\pi n)}\right)\right)^{-2h_+}.\end{aligned} \tag{5.6}$$

[5] We evaluate $\frac{d\lambda'_+}{du'_+}$ at $r=\infty$ since Φ_0 is a conformal field at the boundary.

5.2 One-point function

A first aspect of the holographic correspondence that we want to stress is the following. The one-point function for the energy momentum tensor in these backgrounds was computed in [19] on the basis of the supergravity solution. The one-point function has to agree with the vacuum expectation value of the boundary energy momentum tensor [46]. The vacuum expectation value of the energy momentum tensor in the boundary theory was computed in [10]. The computation was based on performing a Bogoliubov transformation on the Poincare vacuum, taking the vacuum to a thermally excited state in the CFT, for left- and rightmovers in the generic BTZ background, and for (say) leftmovers only for the extremal BTZ background. It is a trivial but important observation that indeed these two computations agree.

The second aspect we stress here is that the Poincare vacuum does not undergo a Bogoliubov transformation for the massless black hole. Indeed, the massless black hole is described, precisely by the group element parametrized in Poincare coordinates. In other words, the Poincare vacuum in the boundary CFT agrees with the vacuum for the massless BTZ black hole background (up to the global angular identification $\phi \sim 2\pi n$). We stress this because the standard association of the results in Poincare coordinates to the AdS_3 background might be cause of confusion in our context.

5.3 Two-point function

Using the bulk-to-boundary propagator, we can compute two- and three-point functions in the boundary conformal field theory in a supergravity approximation. The two-point functions for operators O, O' in the boundary theory are (see e.g. [19, 45]):

$$\langle OO' \rangle \sim (4\pi^2 T_+ T_-)^{2h_+} \sum_{n=\infty}^{-\infty} \sinh^{-2h_+}(\pi T_+(\Delta u_+ + 2\pi n)) \sinh^{-2h_+}(\pi T_-(\Delta u_- + 2\pi n)) \,, \tag{5.7}$$

and, as remarked before, they allow smooth limits in the extremal cases. It was argued in [19] that these can be recuperated from the two-point function in the full string theory. Indeed, in the AdS_3 context, it is known that the boundary correlation functions are obtained by integrating worldsheet correlation functions over the worldsheet.

5.4 Three-point function

What we want to point out here is the relation to the computation of the three-point functions in a supergravity limit in euclidean signature, and to show that our results can be used to define a lorentzian generalisation. (Note that this remark also applies to the two-point overlaps with some care about normalisations.) The computation of the three-point function in euclidean signature in a supergravity approximation [47] was done in Poincare coordinates, in coordinate space.[6] We would like to compare these boundary conformal field theory three-point functions in a supergravity approximation with our results for the pointparticle limit of the worldsheet three-point function.

[6]Stricly speaking, once again, we need to sum over images to take into account the periodicity of the ϕ-coordinate.

First of all, we note the relation between the bulk-to-boundary propagator in coordinate and momentum space. Using integrals noted in the appendix (and, in particular, performing the angular integral first), we can prove the following two-dimensional Fourier-transform (i.e. Fourier transform on the boundary):

$$\int e^{-ikx} dx \frac{\Delta - 1}{\pi} \left(\frac{z_0}{z_0^2 + |x|^2} \right)^{\Delta} = \frac{2^{-\nu+1}}{\Gamma(\nu)} z_0 k^{\nu} K_{\nu}(kz_0) \tag{5.8}$$

where $\nu = \Delta - 1$. That coincides with the *naive* propagator in [47] (which has been argued to be ill-suited for computing the two-point function — it misses out on a crucial normalisation factor necessary for verification of the Ward identities). Thus, the Fourier transform of the Poincare bulk-to-boundary propagator is the modified Bessel function. It should be clear now that the three-point function $A_1(x_i)$ in [47] is related, via FT on the boundary, to the integral over three modified Bessel functions. More explicitly, we should have that

$$\begin{aligned} A_1(k_i) &= \frac{1}{(2\pi)^3} \int dx_i A_1(x_i) e^{-i \sum k_i x_i} \\ &= \frac{-\delta(\sum k_i)}{2\pi} \int \frac{dw_0}{w_0^3} \prod_{i=1}^{3} K_{\Delta_i}(w_0, k_i) \\ &= \frac{-4\delta(\sum k_i)}{\pi} \int dw_0 \prod_{i=1}^{3} \left(2^{\nu_i}\Gamma(\nu_i)\right)^{-1} k_i^{\nu_i} K_{\nu_i}(k_i w_0) \,. \end{aligned} \tag{5.9}$$

The second line contains the propagator K_{Δ_i} in momentum space, the third line modified Bessel functions K_{ν_i}. The last line is the integral over three modified Bessel functions that we encountered as the triple overlap of certain particle wavefunctions in the massless black hole background.

In a general AdS/CFT context, analysed in Poincare coordinates, in lorentzian signature, we are free to choose a different propagator, with extra terms proportional to ordinary Bessel functions [43]. This corresponds to choosing a different coherent state in the boundary conformal field theory. If we want to compute the three-point function in this lorentzian conformal field theory, we would evaluate the integrals over a product of three Bessel functions in the bulk. Thus, the integrals over three Bessel functions that we provided in section 4 and the appendix allow for a computation of the three-point functions in lorentzian signature with any given choice of propagator.[7]

Note that in momentum space, we only needed to restrict $\mu - \lambda$ to integers to implement the angular identification for the BTZ black hole, and did not have to deal with the infinite sum over images. This is a definite advantage of working in momentum space.

6. Conclusions and discussion

We analysed the minisuperspace limit of the $\mathrm{SL}(2,\mathbb{R})$ Wess-Zumino-Witten model, in a scheme appropriate for the description of string propagation on the massless BTZ black

[7] In a general AdS/CFT context, the indices ν_i do not satisfy (A.11). One may then define the integral outside the region (A.11) by analytic continuation.

hole background. In particular, we discussed the appropriate Hilbert space (and string spectrum), the group theoretic structure of the three-point functions, and we analyzed some features that are typical of the lorentzian signature of the background.

One goal to keep in mind is the connection by spectral flow between the holographic dual boundary conformal field theory to the theory dual to the AdS_3 background. It would be very useful to work out these correspondences in some detail, since it might shed light on the difficult issue of time-dependent (and possibly singular or unstable) backgrounds in string theory. To that end, it would be instructive to analyse further the boundary conformal algebra, including its supersymmetric extensions, in a string theoretic context, and to connect it via spectral flow in the boundary theory to the analysis performed in the AdS_3 case. It is also important to work out the four-point functions in the particle model, and to connect it to the full four-point function in the (euclidean) conformal field theory. That should enable one to analyze whether the instability found in the parabolic orbifold persists in this background, and whether there is a sensible interpretation of the instability in this context, perhaps via the boundary conformal field theory. Note that the existence of the boundary conformal field theory may be associated directly to the fact that the background is asymptotically AdS_3 (and not Minkowski space).

There are other issues that are important to address. One is the extension of our analysis to other BTZ black hole backgrounds. That will involve a substantial amount of $\mathrm{SL}(2,\mathbb{R})$ group theory. It is important, we believe, to do at least part of the analysis in the lorentzian signature spacetime, to get a better understanding of time-dependent features of the AdS/CFT correspondence, and to be able to extract time-dependent physics from our knowledge of the euclidean $\mathrm{SL}(2,\mathbb{C})/\,\mathrm{SU}(2)$ CFT. Moreover, it would be interesting to reinterpret our results on three-point functions in a more general lorentzian AdS/CFT context.

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A. Integrals

A.1 Fourier transforms

We note a few useful integral formulas:

$$\begin{aligned}
\int_0^{2\pi} d\phi e^{ia\cos\phi} &= 2\pi J_0(a) \\
\int_0^{\infty} x^{\mu} J_\nu(ax)dx &= 2^{\mu} a^{-\mu-1}\frac{\Gamma\left(\frac{1+\nu+\mu}{2}\right)}{\Gamma\left(\frac{1+\nu-\mu}{2}\right)} \\
\int_0^{\infty} x^{\nu+1} J_\nu(bx)\left(x^2+a^2\right)^{-\mu-1} dx &= \frac{a^{\nu-\mu} b^{\mu}}{2^{\mu}\Gamma(\mu+1)} K_{\nu-\mu}(ab)\,. \qquad \text{(A.1)}
\end{aligned}$$

The second formula is valid when $-\operatorname{Re}(\nu)-1<\operatorname{Re}(\mu)<1/2$, $a>0$ and the third when $-1<\operatorname{Re}(\nu)<\operatorname{Re}(2\mu+\frac{3}{2})$, $a,b>0$. We also have:

$$|x|^{-2\nu-2} = \frac{1}{\pi}2^{-2\nu-2}\frac{\Gamma(-\nu)}{\Gamma(\nu+1)}\int dk e^{ik.x}|k|^{2\nu} \tag{A.2}$$

which agrees with the appendix of [47].

A.2 Integrals of Bessel functions

$$(J_\mu,J_\nu) \equiv \int_0^\infty \frac{dx}{x}J_\mu(ax)J_\nu(ax) = \frac{2}{\pi}\frac{1}{\mu^2-\nu^2}\sin\frac{\pi}{2}(\mu-\nu)\,, \tag{A.3}$$

which is valid for

$$\operatorname{Re}(\mu+\nu)>0\,, \qquad a>0\,. \tag{A.4}$$

Next we have ([39], p. 385):

$$\begin{aligned}(K_\mu,K_\nu)_{b,\alpha} &\equiv \int_0^\infty dx\, x^{\alpha-1}K_\mu(bx)K_\nu(bx) \\ &= \frac{2^{\alpha-3}}{b^\alpha\Gamma(\alpha)}\Gamma\left(\frac{\alpha+\mu+\nu}{2}\right)\Gamma\left(\frac{\alpha+\mu-\nu}{2}\right)\Gamma\left(\frac{\alpha-\mu+\nu}{2}\right)\Gamma\left(\frac{\alpha-\mu-\nu}{2}\right),\end{aligned} \tag{A.5}$$

which is valid for

$$\operatorname{Re}b>0\,, \qquad \operatorname{Re}\alpha>|\operatorname{Re}\mu|+|\operatorname{Re}\nu|\,. \tag{A.6}$$

Here we list formulas of integrals of three Bessel functions. We use the notation $\Gamma\left[{a_1,a_2,\ldots \atop b_1,b_2,\ldots}\right]=\frac{\Gamma(a_1)\Gamma(a_2)\ldots}{\Gamma(b_1)\Gamma(b_2)\ldots}$.

$$\begin{aligned}\int_0^\infty dx\, x^{\rho-1}J_\lambda(ax)J_\mu(bx)J_\nu(cx) &= \frac{2^{\rho-1}a^\lambda b^\mu}{c^{\rho+\lambda+\mu}}\Gamma\left[\frac{\rho+\lambda+\mu+\nu}{2}\right]\lambda+1,\mu+1,1-\frac{\rho+\lambda+\mu-\nu}{2}\times \\ &\quad\times F_4\left(\frac{\rho+\lambda+\mu-\nu}{2},\frac{\rho+\lambda+\mu+\nu}{2};\lambda+1,\mu+1;\frac{a^2}{c^2},\frac{b^2}{c^2}\right), \qquad (\text{A.7}) \\ \int_0^\infty dx\, x^{\rho-1}J_\lambda(ax)J_\mu(bx)K_\nu(cx) &= \frac{2^{\rho-2}a^\lambda b^\mu}{c^{\rho+\lambda+\mu}}\Gamma\left[{\frac{\rho+\lambda+\mu+\nu}{2},\frac{\rho+\lambda+\mu-\nu}{2} \atop \lambda+1,\mu+1}\right]\times \qquad (\text{A.8}) \\ &\quad\times F_4\left(\frac{\rho+\lambda+\mu-\nu}{2},\frac{\rho+\lambda+\mu+\nu}{2};\lambda+1,\mu+1;-\frac{a^2}{c^2},-\frac{b^2}{c^2}\right), \\ \int_0^\infty dx\, x^{\rho-1}J_\lambda(ax)K_\mu(bx)K_\nu(cx) &= \frac{2^{\rho-3}a^\lambda}{c^{\rho+\lambda}\Gamma(\lambda+1)}[A(\mu)+A(-\mu)]\,, \qquad (\text{A.9}) \\ A(\mu) &= \left(\frac{b}{c}\right)^\mu\Gamma\Big[-\mu,\frac{\rho+\lambda+\mu+\nu}{2},\frac{\rho+\lambda+\mu-\nu}{2}\Big]\times \\ &\quad\times F_4\left(\frac{\rho+\lambda+\mu-\nu}{2},\frac{\rho+\lambda+\mu+\nu}{2};\lambda+1,\mu+1;-\frac{a^2}{c^2},\frac{b^2}{c^2}\right), \\ \int_0^\infty dx\, x^{\rho-1}K_\lambda(ax)K_\mu(bx)K_\nu(cx) &= \frac{2^{\rho-4}}{c^\rho}\sum_{\sigma_1,\sigma_2=\pm1}A(\sigma_1\lambda,\sigma_2\mu)\,, \\ A(\lambda,\mu) &= \left(\frac{a}{c}\right)^\lambda\left(\frac{b}{c}\right)^\mu\Gamma\Big[-\mu,-\lambda,\frac{\rho+\lambda+\mu+\nu}{2},\frac{\rho+\lambda+\mu-\nu}{2}\Big]\times \qquad (\text{A.10}) \\ &\quad\times F_4\left(\frac{\rho+\lambda+\mu-\nu}{2},\frac{\rho+\lambda+\mu+\nu}{2};\lambda+1,\mu+1;\frac{a^2}{c^2},\frac{b^2}{c^2}\right).\end{aligned}$$

They are found in p. 717 in [48], p. 717 in [48], p. 399 in [39], p. 403 in [39], and valid for

$$\begin{aligned}
&\text{(i)} \quad \mathrm{Re}(\lambda+\mu+\nu+\rho)>0\,; \qquad \mathrm{Re}\,\rho<\frac{5}{2}\,; \qquad a,b>0\,; \qquad c>a+b \\
&\text{(ii)} \quad \mathrm{Re}(\lambda+\mu+\rho)>|\,\mathrm{Re}\,\nu|\,; \qquad \mathrm{Re}\,c>|\,\mathrm{Im}\,a|+|\,\mathrm{Im}\,b| \\
&\text{(iii)} \quad \mathrm{Re}(\lambda+\rho)>|\,\mathrm{Re}\,\mu|+|\,\mathrm{Re}\,\nu|\,; \qquad \mathrm{Re}(b+c)>|\,\mathrm{Im}\,a| \\
&\text{(iv)} \quad \mathrm{Re}\,\rho>|\,\mathrm{Re}\,\lambda|+|\,\mathrm{Re}\,\mu|+|\,\mathrm{Re}\,\nu|\,; \qquad \mathrm{Re}(a+b+c)>0
\end{aligned} \tag{A.11}$$

respectively. The formulas (A.8)–(A.10) may be obtained from the first formula (A.7) by using the definition of K_μ in terms of J_μ. The first formula may be obtained by expanding the first two Bessel functions, and computing the moments of the third Bessel functions term by term.

For $\rho=1$, there is a formula (p. 228 in [39]),

$$\begin{aligned}
\int_0^\infty dx\, J_\lambda(ax)J_\mu(bx)J_\nu(cx) \;=\; & \frac{a^\lambda b^\mu}{c^{1+\lambda+\mu}}\Gamma\left[\begin{matrix}\frac{\lambda+\mu+\nu+1}{2} \\ \end{matrix}\right]\lambda+1,\mu+1,\frac{1+\nu-\lambda-\mu}{2}\times \\
& \times {}_2F_1\left(\frac{1+\lambda+\mu-\nu}{2},\frac{1+\lambda+\mu+\nu}{2};\lambda+1;z_+\right)\times \\
& \times {}_2F_1\left(\frac{1+\lambda+\mu-\nu}{2},\frac{1+\lambda+\mu+\nu}{2};\mu+1;z_-\right),
\end{aligned} \tag{A.12}$$

where

$$\begin{aligned}
&\mathrm{Re}(\lambda+\mu+\nu)>-1\,; \qquad a,b>0\,; \qquad c>a+b\,; \\
&2c^2 z_\pm = \pm a^2 \mp b^2 + c^2 + \sqrt{(a^2+b^2-c^2)^2-4a^2b^2}\,.
\end{aligned} \tag{A.13}$$

This agrees with the identity between F_4 and ${}_2F_1$ in [49].

B. Parabolic CGC

We rewrite the integrand for $C_\omega(\lambda)$ using the following Fourier transform:

$$\begin{aligned}
|x_i-x_j|^{-2a_k}\,\mathrm{sign}^{2\eta_k}(x_i-x_j) \;=\; & \frac{\cos\left(\frac{\pi}{2}(-2a_k+1+2\eta_k)\right)}{\pi}\Gamma(-2a_k+1)\times \\
& \times\int_{-\infty}^{+\infty}(-isgn\xi)^{2\eta_k}|\xi|^{2a_k-1}e^{-i\xi(x_i-x_j)}d\xi\,.
\end{aligned} \tag{B.1}$$

After performing the integrals over x_i that yield three delta-functions, we obtain:

$$C_\omega(\lambda) \;=\; 8\kappa\delta(\lambda_1+\lambda_2-\lambda_3)\prod_{i=1}^{3}\Gamma(-2a_i+1)\sin\left(\frac{\pi}{2}(2a_i-2\eta_i)\right)(-i)^{2\eta_i}I'_\lambda \tag{B.2}$$

where

$$I'_\lambda = \int_{-\infty}^{+\infty} d\xi\,\mathrm{sign}^{2\eta_1}(\lambda_2+\xi)\,\mathrm{sign}^{2\eta_2}(-\lambda_1+\xi)\,\mathrm{sign}^{2\eta_3}(\xi)|\lambda_2+\xi|^{2a_1-1}|-\lambda_1+\xi|^{2a_2-1}|\xi|^{2a_3-1}\,. \tag{B.3}$$

(Note at this point that the integral is invariant under the exchange of $\lambda_{1,2}$, $\eta_{1,2}$ and $a_{1,2}$ up to a factor of $(-1)^{2\sum\eta_i}$. We evaluate the integral I'_λ in the case where $-\lambda_2 < 0 < \lambda_1$, by dividing the real line into four regions, separated by the points $-\lambda_2, 0, \lambda_1$. The four regions give rise to the four terms:

$$\begin{aligned}
I'_\lambda &= (-1)^{2\eta_1+2\eta_2+2\eta_3}\lambda_2^{2a_1+2a_2+2a_3-2}B(-2a_1-2a_2-2a_3+2, 2a_1)\times \\
&\quad\times {}_2F_1\left(-2a_2+1, -2a_1-2a_2-2a_3+2; -2a_2-2a_3+2; -\frac{\lambda_1}{\lambda_2}\right)+ \\
&\quad+(-1)^{2\eta_2+2\eta_3}\lambda_1^{2a_2-1}\lambda_2^{2a_1+2a_3-1}B(2a_1,2a_3)\,{}_2F_1\left(-2a_2+1, 2a_3; 2a_1+2a_3; -\frac{\lambda_2}{\lambda_1}\right)+ \\
&\quad+(-1)^{2\eta_2}\lambda_2^{2a_1-1}\lambda_1^{2a_2+2a_3-1}B(2a_2,2a_3)\,{}_2F_1\left(-2a_1+1, 2a_3; 2a_2+2a_3; -\frac{\lambda_1}{\lambda_2}\right)+ \\
&\quad+\lambda_1^{2a_1+2a_2+2a_3-2}B(-2a_1-2a_2-2a_3+2, 2a_2)\times \\
&\quad\times {}_2F_1\left(-2a_1+1, -2a_1-2a_2-2a_3+2; -2a_1-2a_3+2; -\frac{\lambda_2}{\lambda_1}\right).
\end{aligned} \tag{B.4}$$

Then we use the following relation to rewrite the expression in terms of hypergeometric functions with the argument $-\lambda_1/\lambda_2$):

$$\begin{aligned}
(-z)^{-a}\,{}_2F_1\left(a, a+1-c; a+1-b; z^{-1}\right) &= \Gamma\begin{bmatrix}1-c, a+1-b\\ 1-b, a+1-c\end{bmatrix}{}_2F_1(a,b;c;z)- \\
&\quad-\Gamma\begin{bmatrix}c, 1-c, a+1-b\\ 2-c, c-b, a\end{bmatrix}e^{i\pi(c-1)}z^{1-c}\times \\
&\quad\times {}_2F_1(1+a-c, b+1-c; 2-c; z).
\end{aligned} \tag{B.5}$$

The resulting expression, after using Γ-function identities is:

$$\begin{aligned}
I'_\lambda &= \lambda_2^{2a_1+2a_2+2a_3-2}\times \\
&\quad\times\left\{{}_2F_1\left(-2a_2+1, -2a_1-2a_2-2a_3+2; -2a_2-2a_3+2; -\frac{\lambda_1}{\lambda_2}\right)\times\right. \\
&\quad\times\Gamma\begin{bmatrix}2a_1, -2a_1-2a_2-2a_3+2\\ -2a_2-2a_3+2\end{bmatrix}\times \\
&\quad\times\frac{(-1)^{2\eta_1+2\eta_2+2\eta_3}\sin\pi(2a_2+2a_3)+(-1)^{2\eta_2+2\eta_3}\sin\pi(2a_1+2a_2+2a_3)-\sin\pi 2a_1}{\sin\pi(2a_2+2a_3)}+ \\
&\quad+\left(\frac{\lambda_1}{\lambda_2}\right)^{2a_2+2a_3-1}{}_2F_1\left(-2a_1+1, 2a_3; 2a_2+2a_3; -\frac{\lambda_1}{\lambda_2}\right)\Gamma\begin{bmatrix}2a_2, 2a_3\\ 2a_2+2a_3\end{bmatrix}\times \\
&\quad\left.\times\frac{(-1)^{2\eta_2}\sin\pi(2a_2+2a_3)+(-1)^{2\eta_2+2\eta_3}\sin\pi 2a_2+\sin\pi 2a_3}{\sin\pi(2a_2+2a_3)}\right\}.
\end{aligned} \tag{B.6}$$

Now that we evaluated I'_λ, we can evaluate our original $C_\omega(\lambda)$, the CGC's in the parabolic basis. We plug in the values for the a_i (in some instances) and η_i, and rewrite the expression a little to obtain ($2\tau_i + 1 = \nu_i$):

$$
\begin{aligned}
C_\omega(\lambda) \;=\; & 8\kappa\delta(\lambda_1+\lambda_2-\lambda_3)\Gamma\left(\frac{1-\nu_1+\nu_2-\nu_3}{2}\right)\Gamma\left(\frac{1+\nu_1-\nu_2-\nu_3}{2}\right)\times \\
& \times\Gamma\left(\frac{1+\nu_1+\nu_2+\nu_3}{2}\right)\prod_i \sin(\pi(a_i-\eta_i))(-i)^{2\eta_i}\lambda_2^{\frac{-\nu_1-\nu_2+\nu_3-1}{2}}\times \\
& \times\Bigg\{ {}_2F_1\left(\frac{\nu_1-\nu_2-\nu_3+1}{2},\frac{\nu_1+\nu_2-\nu_3+1}{2};\nu_1+1;-\frac{\lambda_1}{\lambda_2}\right)\times \\
& \times\Gamma\begin{bmatrix}\frac{\nu_1-\nu_2+\nu_3+1}{2},\frac{\nu_1+\nu_2-\nu_3+1}{2} \\ \nu_1+1\end{bmatrix}\times \\
& \times\left((-1)^{2\omega}-\frac{(-1)^{2\epsilon_1}\cos\frac{\pi}{2}(\nu_1+\nu_2-\nu_3)+\cos\frac{\pi}{2}(\nu_1-\nu_2+\nu_3)}{\sin\pi\nu_1}\right)+ \\
& +\left(\frac{\lambda_1}{\lambda_2}\right)^{-\nu_1}{}_2F_1\left(\frac{-\nu_1+\nu_2-\nu_3+1}{2},\frac{-\nu_1-\nu_2-\nu_3+1}{2};-\nu_1+1;-\frac{\lambda_1}{\lambda_2}\right)\times \\
& \times\Gamma\begin{bmatrix}\frac{-\nu_1+\nu_2+\nu_3+1}{2},\frac{-\nu_1-\nu_2-\nu_3+1}{2} \\ \end{bmatrix}-\nu_1+1\times \\
& \times\left((-1)^{2\omega+2\epsilon_2}+\frac{(-1)^{2\epsilon_1}\cos\frac{\pi}{2}(-\nu_1+\nu_2+\nu_3)+\cos\frac{\pi}{2}(\nu_1+\nu_2+\nu_3)}{\sin\pi\nu_1}\right)\Bigg\}, \qquad \text{(B.7)}
\end{aligned}
$$

which can be further simplified a bit using trigonometry.

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